CATASTROPHE IN CAPITAL ACCUMULATION
—Structural stability with overall instability—

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1. Introduction

In our daily life we often experience the discontinuities of phenomenon such as a sudden depression of the economy, a sudden increase of unemployment, a sudden rise of prices, etc.. So far these sudden changes of phenomena cannot be treated completely because of the limitation of mathematical language which is based on the concepts of continuity. In his noteworthy literature, Thom writes, "and nothing disturbs a mathematician more than discontinuity, since all applicable quantitative models depend on the use of analytic and therefore continuous functions."(1)

To overcome these limitations, he himself has presented in it the static and metabolic models which enable to explain the discontinuities or a morphogenesis in the nature of the system(2). These new methods are now called the theory of catastrophe(3), bringing a revolutionary tool of mathematical language to the fields of study such as biology and social sciences where discontinuous phenomena are usually observed.

The idea of this paper comes from the metabolic model. Its purpose is to construct a simple economic model which gives one of the explanations to the discontinuous phenomenon in capital accumulation in our economy.

2. A model of capital accumulation

First of all, we begin with the construction of our model in the economy(4). Let K be the existing capital stock, and α a capital coefficient. Then, we have the output;

(1) $Y = \frac{1}{\alpha} K$

Some parts of this output are saved, increasing the capital stock by $\dot{K}$;

(2) $\dot{K} = sY$

where s denotes a social rate of saving, $0 < s < 1$.

If the investment does not become equal to the increment of capital stock, then disequilibrium takes place in the goods market. Therefore, we assume here that the investment is always equal to the saving;

(3) $I = \dot{K}$

In order for this to be possible, α must work as an adjustable variable. Hence $\frac{1}{\alpha}$ is interpreted as a capacity level of capital stock. How, then, is the investment decided? It is reasonable to think that entrepreneurs usually decide the rate of capital accumulation, not the absolute value of investment; that is,

(4) $\frac{I}{K} = g$

Then, how is this rate decided? We assume, further, that entrepreneurs' attitude to invest is dependent on two factors; an economic factor and an external factor of economic system.

Let α^* be the normal capital coefficient entrepreneurs become satisfied with. If the present capital coefficient is less than the normal one ($\alpha^* > \alpha$), then they feel a deficiency of capital stock and wish to increase it, so that the rate of capital accumulation grows rapidly. Conversely, if $\alpha^* < \alpha$, they don't want to increase the rate at all. It is possible to conclude from these reasoning that the rate of capital accumulation primarily depends on the difference between the normal and actual capital coefficients. This effect is what we have called the economic factor, which can be regarded as a strong incentive for entrepreneurs to invest capital stock or to withdraw from it.

In our economy government plays a very important role. For example, monetary policies often bring a very strong influence on the decisions of entrepreneurs to invest. Accordingly, they are always forced to behave within the limitations of these externally given situations. At a time of accumulative boom, government's policy of squeezing money as an anti-inflation action will usually discourage them fron increasing the capital accumulation furthermore. Vice-versa in a time of depression. We shall call this effect the external factor so long as it influences their attitude of investment.

From these we can reasonably formulate the accumulation function, being assumed to be analytic, as follows;

(5) $\dot{g} = \phi(\alpha^* - \alpha) + \theta$

where $\phi(0) = 0$, $\phi' > 0$ and $\phi'' < 0$.

The implication of the assumption $\phi'' < 0$ is that as g increases, entrepreneurs want to accumulate more but less than the previous rate. Now our model becomes complete, consisting of five equations with five unknowns K, Y, α, I, g and two parameters s, θ.

3. Structural stability with overall instability

We obtain the following from (1) through (4);

(6) $g\alpha = s$

which can be expressed as

(7) $\alpha = \alpha(g, s)$

Then, (5) becomes

(8) $\dot{g} = \phi(\alpha^* - \alpha(g, s)) + \theta$

ϕ being an analytic function,(8) can be rewritten as follows by the Taylor's formula, ignoring the terms of degree more than two,

$$(9)\ \dot{g} = -\phi' \frac{\partial \alpha}{\partial g}\bigg|_{g=g^*} (g-g^*) - \frac{1}{2}\left[-\phi''\left(\frac{\partial \alpha}{\partial g}\right)^2 + \phi' \frac{\partial^2 \alpha}{\partial g^2}\right]_{g=g^*} (g-g^*)^2 + \theta$$

$$= \phi' \frac{s}{g^{*2}} (g-g^*) - \frac{1}{2}\left[-\phi'' \frac{s^2}{g^{*4}} + \phi' \frac{2s}{g^{*3}}\right] (g-g^*)^2 + \theta\ ,$$

Putting $x = g-g^*$, $\lambda(s) = \phi' \frac{s}{g^{*2}} > 0$, and $\mu(s) = -\frac{1}{2}\left[-\phi'' \frac{s}{g^{*4}} + \phi' \frac{2s}{g^{*3}}\right] < 0$

,we have

(10) $\dot{x} = \mu(s)x^2 + \lambda(s)x + \theta$

Let Π be a mapping, Π: $X \times R^2 \to X$ such that

(11) $\Pi(x, s, \theta) = \mu(s)x^2 + \lambda(s)x + \theta$

where X is a space of an internal variable or a configuration space and R^2 is a space of external parameters.

Next we define a vector field Π on X, depending on a point (s, θ) in R^2, by

(12) $\Pi_{s,\theta}(x) = \Pi(x, s, \theta)$

Then, a dynamical system (X, Π) can be defined by a vector field Π on X.

Our problem to be solved is to classify every vector field Π on X into structurally stable vector fields and structurally unstable ones, and to give them economic interpretations. Here we mean by "structurally stable" the following stability.

Definition 1: A dynamical system Π on X is structurally stable if a sufficiently small (in the C^1-norm) perturbation of the vector field Π does not alter the qualitative nature of the system, where Π on X is given the topology defined by the metric (5)

(13) $d(\Pi_0, \Pi_1) = \sup\limits_{X} \left\{ |\Pi_0 - \Pi_1| + \left| \frac{d\Pi_0}{dx} - \frac{d\Pi_1}{dx} \right| \right\}$.

It is convenient for the classification of the vector fields to consider, first of all, the following five cases one by one according to the value of θ.

Case 1: $\theta = 0$. From (11) and (12), we have

(14) $\Pi_{s,o}(x) = x(\mu(s)x + \lambda(s))$

Let the zeros of $\Pi_{s,o}(x)$ be x_1, x_2 $(x_1 \leq x_2)$, respectively. Then

(15) $x_1 = 0$ and $x_2 = \omega(s)$

where $\omega(s) = -\frac{\lambda(s)}{\mu(s)} = \frac{2\phi' g^{*2}}{2\phi' g^* - \phi'' s} > 0$

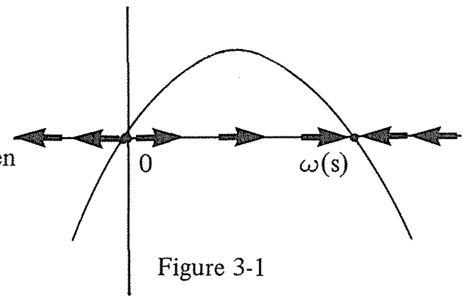

Figure 3-1

The vector field is shown in Figure 3-1, where the origin is a repellor and $\omega(s)$ is an attractor.

Case 2: $\theta > 0$. In this case we have the following relations;

(16) $x_1 + x_2 = \omega(s) > 0, \; x_1 x_2 = \frac{\theta}{\mu(s)} < 0$

(17) $D_{\Pi} = \lambda(s)^2 - 4\theta\mu(s) > 0$

where D_{Π} denotes a discriminant of the equation $\Pi_{s,\theta}(x) = 0$.

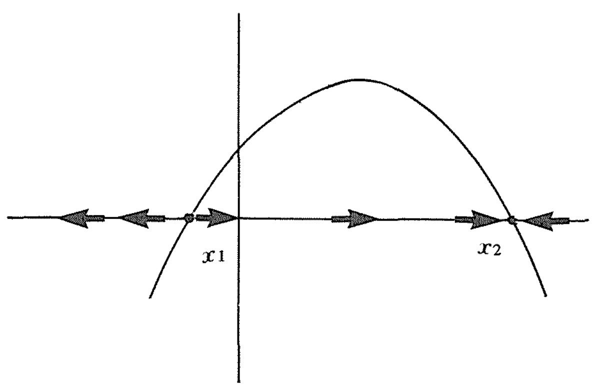

Figure 3-2

Accordingly x_1 takes a negagive value and x_2 a positive one. Figure 3-2 shows the vector field in this case where x_1 is a repellor and x_2 an attractor.

Case 3: $-\frac{\omega(s)\lambda(s)}{4} < \theta < 0$. Here we have

(18) $x_1 + x_2 = \omega(s) > 0, \; x_1 x_2 = \frac{\theta}{\mu(s)} > 0$

(19) $D_{\Pi} = \lambda(s)^2 - 4\theta\mu(s) > \lambda(s)^2 + \omega(s)\lambda(s)\mu(s) = 0$

Thus, both x_1 and x_2 have positive values. The vector field is shown in Figure 3-3 where x_1 is a repellor and x_2 an attractor.

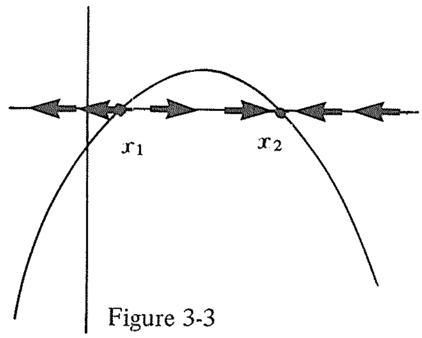

Figure 3-3

Case 4: $\theta = -\frac{\omega(s)\lambda(s)}{4}$. Here we obtain

(20) $x_1 + x_2 = \omega(s)$, $x_1 x_2 = \left(\frac{\omega(s)}{2}\right)^2$

(21) $D_{\Pi} = \lambda(s)^2 - 4\theta\mu(s) = \lambda(s)^2 + \omega(s)\lambda(s)\mu(s) = 0$

,which imply that $x_1 = x_2 = \frac{\omega(s)}{2} > 0$.

In this case, both an attractor and a repellor disappear from the vector field as drawn in Figure 3-4.

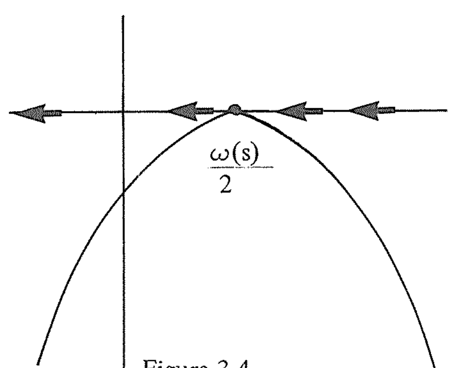

Figure 3-4

Case 5: $\theta < -\frac{\omega(s)\lambda(s)}{4}$. We have

(22) $D_{\Pi} = \lambda(s)^2 - 4\theta\mu(s) < \lambda(s)^2 + \omega(s)\lambda(s)\mu(s) = 0$

,which implies that there exist no zeros in the equation $\Pi_{s,\theta}(x) = 0$ so that every flow on the vector field streams to the left as shown in Figure 3-5.

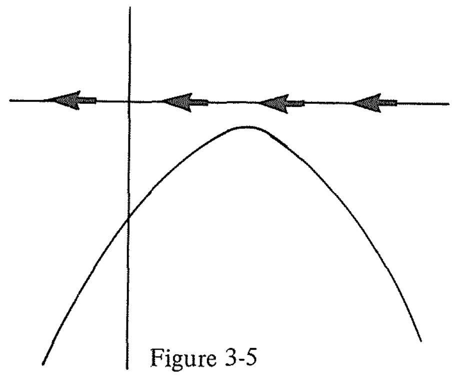

Figure 3-5

From these analyses, we can easily conclude the following three points; (1) the vector fields with an attractor and a repellor in the case 1, 2 and 3 are structurally stable, (2) the vector field without any attractor and repellor in the case 5 is also structurally stable, (3) the vector field in the case 4 is structurally unstable. In other words, we cannot find any other vector field in case 4 in the neighborhood of, say, Π_0 in the sense of C^1-norm defined in (13) which does not alter its qualitative structure. For example, let $\Pi_0 = \Pi(x, s, \theta_0)$, $\Pi_1 = \Pi(x, s, \theta_1)$, respectively and Π_1 be in the ε-neighborhood of Π_0, being different from Π_0; that is,

$$(23)\quad 0 < d(\Pi_0, \Pi_1) = \sup_X \left\{ \left| \Pi_0 - \Pi_1 \right| + \left| \frac{d\Pi_0}{dx} - \frac{d\Pi_1}{dx} \right| \right\} = |\theta_0 - \theta_1| < \varepsilon,$$

,then Π_1 clearly belongs to the vector field in case 3 when $\theta_0 < \theta_1 < \theta_0 + \varepsilon$ or case 5 when $\theta_0 - \varepsilon < \theta_1 < \theta_0$, so that Π_0 and Π_1 are entirely different in structure. Consequently Π_0 becomes structurally unstable and called a bifurcation for this reason. Vector fields can be said to change their structures of flows suddenly at $\theta_0 = -\frac{\omega(s)\lambda(s)}{4}$ Thus θ_0 becomes a critical point. This process is vividly illustrated in Figure 3-6 and Figure 3-7.

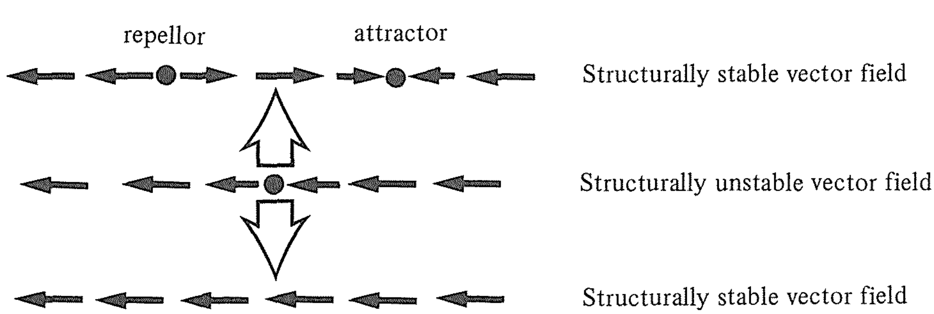

Figure 3-6

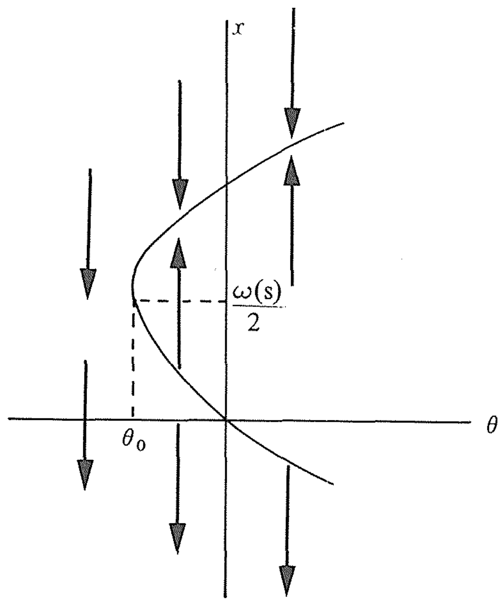

Figure 3-7

Figure 3-7 is obtained from the following equation:

(24) $\theta = -\mu(s)x^2 - \lambda(s)x$

$$= -\mu(s)\left(x - \frac{\omega(s)}{2}\right)^2 - \frac{\omega(s)\lambda(s)}{4}$$

So far vector fields are considered from the side of θ only, s being assumed to be constant for convenience, since s does not influence so much as compared with θ . But it is necessary to examine the working of s to consider our model completely.

When s runs from zero to one, a vertex $\left(\frac{\omega(s)}{2}, -\frac{\omega(s)\lambda(s)}{4}\right)$ in Figure 3-7 moves from v_1 to v_2 where

(25) $v_1 = \left(\frac{\omega(0)}{2}, -\frac{\omega(0)\lambda(0)}{4}\right) = \left(\frac{g^*}{2}, 0\right)$

$$v_2 = \left(\frac{\omega(1)}{2}, -\frac{\omega(1)\lambda(1)}{4}\right) = \left(\frac{\phi' g^{*2}}{2\phi' g^* - \phi''}, -\frac{\phi'^2}{2(2\phi' g^* - \phi'')}\right)$$

and $v_1 > v_2$

This is shown in Figure 3-8.

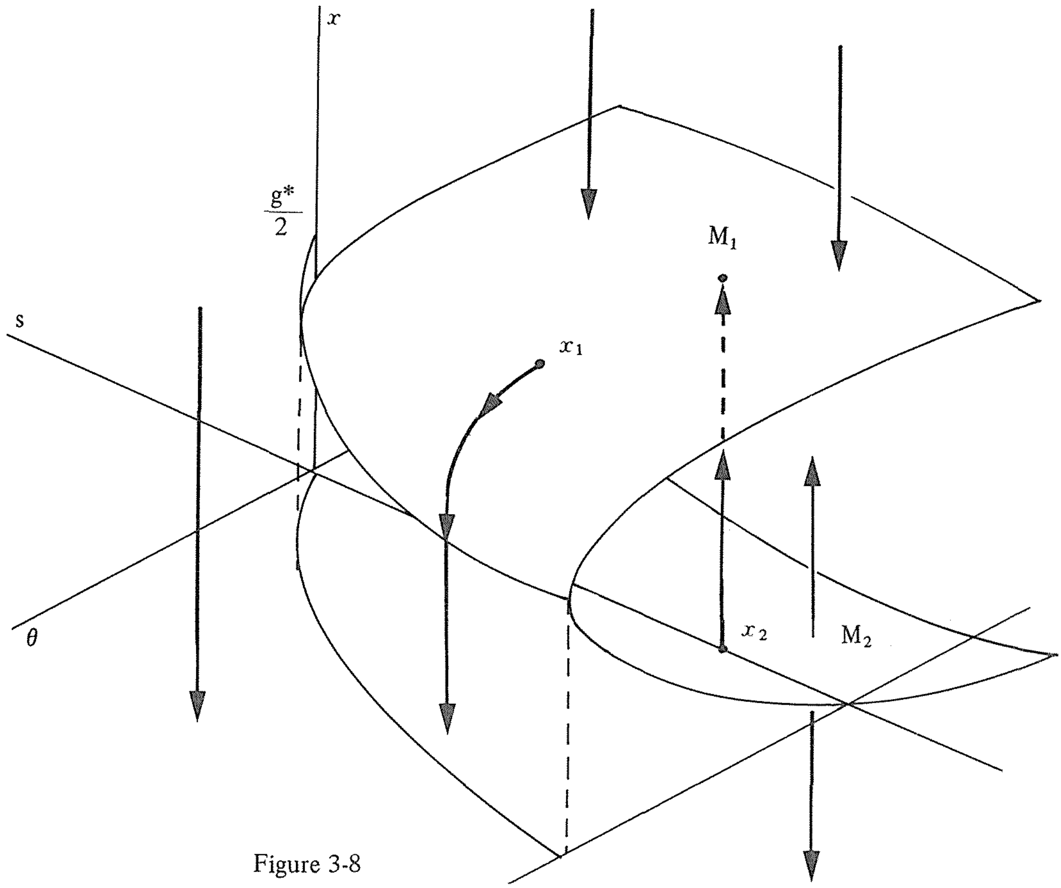

Figure 3-8

In this figure the set of singularities of Π or stationary points constitutes a 2-dimensional manifold in $X \times R^2$. This manifold is further divided by a fold into two parts; the part of attractors (M_1) and that of repellors (M_2). The fold of manifold is described by the following equation;

(26) $\theta = -\dfrac{\omega(s)\lambda(s)}{4}$

What does this equation mean? Here we define a catastrophe set.

Definition 2: Let $h(s, \theta) = \Pi_{s,\theta}$ be a dynamical system corresponding to a point (s, θ) in R^2, and H be the set of all dynamical systems. Let Z be a set of structurally unstable dynamical systems called a bifurcation set. Then a catastrophe set is defined as the points (s, θ) such that $h(s, \theta)$ is in $Z \subset H$.

These relations are shown in the diagram.(6)

$$\begin{array}{ccc} R^2 & \xrightarrow{h} & H \\ \cup & & \cup \\ K & \xrightarrow{h} & Z \\ \| & & \\ h^{-1}(Z) & & \end{array}$$

Diagram

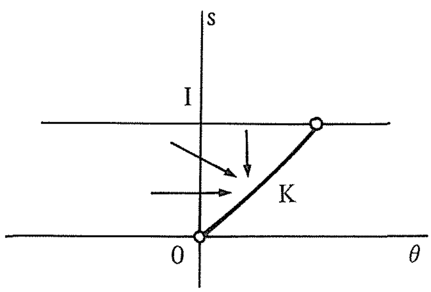

Figure 3-9

In our model, Π_0 is the only point of bifurcation, so that $K=h^{-1}(\Pi_0)$ which is the set satisfying (26). Accordingly a fold is a catastrophe set itself as shown in Figure 3-9, which is obtained by a projection into (s, θ) plane.

We are further guided to define another kind of stability.

Definition 3: Let us consider the system of differential equations;

(27) $\dot{x}=\Pi(x, t)$

We shall separate out of the system a certain solution $x=\phi(t)$ and any other solution determined by the initial condition $x(t_0)$. Then the motion $x=\phi(t)$ is said to be stable in the Lyapunov sense if for every $\varepsilon>0$ and t_0, one can find $\delta(\varepsilon, t_0)>0$ such that, from the inequality $||x(t_0)-\phi(t_0)||<\delta(\varepsilon, t_0)$, there follows the inequality $||x(t)-\phi(t)||<\varepsilon$ for $t\geqq t_0$, where the topology is given by the metric $||x-\phi||=(\sum(x_i-\phi_i)^2)^{\frac{1}{2}}$.(7)

We can easily show from this definition that the manifold consisting of repellors is unstable in this Lyspunov sense. If the initial situation is displaced by a small perturbation from M_2, it does not return to the surface at all. Figure 3-10 is obtained by a projection of M_2 into the (s, θ) plane and shows how the stationary points are overall unstable in the Lyapunov sense with regard to a small perturbation of (s, θ) except the points $\{(s, \theta) \mid 0<s<1, \theta=0\}$ where any change of s cannot repell the initial situation from M_2 as easily derived from (24).

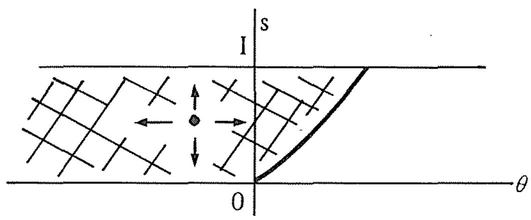

Figure 3-10

Let x_1 be in M_1 as shown in Figure 3-8. x_1 moves about on M_1 according as parameters (s, θ) change When x_1 reaches a point on the fold, it suddenly begin to fall downward, so that it cannot stay on M_1 at all. This implies the structure of the vector field suddenly changes when (s, θ) reaches the catastrophe set K. Next, let x_2 be in M_2. As already shown above, M_2 is the unstable set in the Lyapunov sense. Therefore, x_2 is suddenly thrown away from M_2 with a small perturbation in s and, or θ. If it is thrown upwards, it then converges to M_1. If downwards, it continues to fall. M_2 can be said to be structurally stable to the effect that any small perturbation cannot change the overall instability in the Lyapunov sense.

4. Economic interpretation

Now we shall try to represent an economic interpretation of the above analysis, confining our attention again to the (x, θ) plane for convenience. Considering our economy in this century, we can say that the most dreadful phenomenon has been a business cycle, bringing about a sudden increase of unemployment and the ensuing social chaos. Many economists have been attempting to describe it on a simple canvas. We have assumed here that one of the main reasons it happens is that our economy deeply depends on the arbitrary attitude of entrepreneurs towards the capital accumulation, especially the two factors already stated in section 2 in their decisions of investment, of which the external factor is assumed to play an essential role in a sudden change of capital accumulation. We shall consider this process a little more precisely

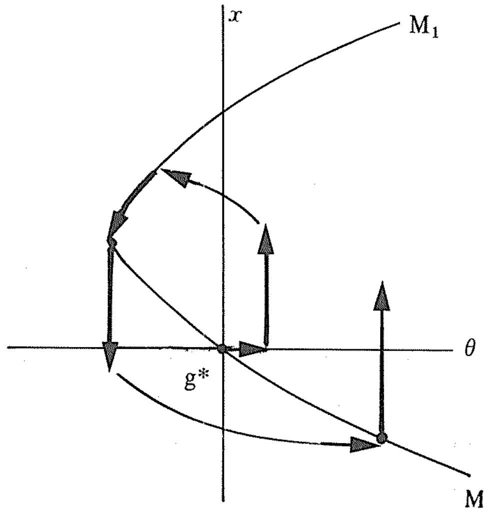

Figure 4-1

with the aid of Figure 4-1. Let us assume at first the economy stays at an origin where $g = g^*$ and $\theta = 0$, so that entrepreneurs are satisfied with a present state of a constant rate of accumulation. They feel neither sufficiency nor deficiency about their capital stock and any external policy by government does not affect their decision at all. Suppose one day the government happens to take a misleading policy of slacking in the money market, for example, and this brings a positive effect among entrepreneurs. Then, by a small perturbation of θ, x will be suddenly thrown from the origin to the right direction and begin to increase, because the origin lies on the unstable surface M_2. This corresponds to the economic situation that, once entrepreneurs feel deficient in their capital stock (when $x = g - g^* > 0$, we have $\alpha^* - \alpha > 0$, so that the actual capital stock is less than the normal one they get satisfied with), they want to increase the capital accumulation furthermore, which will cause a strong excess demand, making them feel deficient in their capital stock more strongly. These situations will continue in an accumulative way. This is an upward phase of a boom. Gradually prices will come to rise and a rapid inflation will begin to suppress our life, so that government will be forced to prevent an over-heated economy with various policies. This will bring a negative effect to the minds of entrepreneurs which will cause x to shift to the left. Sooner or later x will reach M_1 in the second quadrant. Here one should note that x cannot reach M_1 in the first quadrant. Suppose so, then we have, say, $x = \tilde{x} = \tilde{g} - g^* > 0$, $\theta = \tilde{\theta} > 0$ and $\alpha^* - \tilde{\alpha} > 0$. But in this case $\dot{g} = 0 = \phi(\alpha^* - \tilde{\alpha}) + \dot{\theta} > 0$, which is contradiotory.

Since M_1 is a stable surface in the Lyapunov sense, any small perturbation in s and, or θ cannot repell x from there. This is a situation where a constant higher rate, say, $\bar{g}$ of capital accumulation is kept after its attainment than the one with which entrepreneurs get satisfied; that is, $x = \bar{g} - g^* > 0$. They want to accumulate more, but the external situation does not permit to do so, so that they are forced to keep the constant level. But in this situation government will continue to try on removing the excess boom. This will make x shift to the left gradually along the surface M_1. When it comes to a point on a fold, a catastrophe suddenly takes place and it begins to fall downwards. Once it begins to do so it cannot return to the previous situation. The economy jumps into a slump suddenly. Entrepreneurs suffer a tremendous idleness of capital stock and are forced to withdraw investment, which makes a further idleness of capital stock. These situations will continue again in an accumulative way, too. At last government will try to save this situation, bringing a positive effect to their minds once again. Consequently x moves to the right and it will reach M_2 in the fourth quadrant. But as already stated, this surface is unstable so that any small perturbation in s and, or θ will cause the same business cycle as described above. We can say that our economy is always in an unstable situation in the Lyapunov sense which is structurally stable to the effect that any small perturbation does not change this unstable situation.

5. Expanding to the degree three

The above conslusion can be shown to be valid in the neighborhood of the origin even if we consider the higher terms of degree more than two. First of all we consider the case where the function $\phi(\alpha^*-\alpha(g, s))$ is expanded to the degree three.

In this case, (11) is rewritten as

(28) $\Pi(x, s, \theta) = \eta(s)x^3 + \mu(s)x^2 + \lambda(s)x + \theta$

where

(29) $$\eta(s) = -\frac{1}{6}\left[\phi'''\left(\frac{\partial\alpha}{\partial g}\right)^3 - 3\phi''\frac{\partial^2\alpha}{\partial g^2}\ \frac{\partial\alpha}{\partial g} + \phi'\frac{\partial^3\alpha}{\partial g^3}\right] g = g^*$$

$$= \frac{1}{6}\left[\phi'''\frac{s^3}{g^{*6}} - 3\phi''\frac{2s^2}{g^{*5}} + \phi'\frac{6s}{g^{*4}}\right]$$

Let the zeros of $\Pi(x, s, \theta)$ be x_1, x_2, x_3 $(x_1 \leqq x_2 \leqq x_3)$ respectively, and its discriminant be D_Π. Then we have the following relations;

(30) $$\sum_{i=1}^{3} x_i = -\frac{\mu(s)}{\eta(s)}, \qquad \prod_{i<j}^{3} x_i\ x_j = \frac{\lambda(s)}{\eta(s)}, \qquad x_1x_2x_3 = -\frac{\theta}{\eta(s)}$$

where signs of each equation are determined as in Table 5-1, and

(31) $$D_\Pi = \prod_{i<j}^{3}(x_i\ x_j)^2 = \frac{1}{\eta(s)^4}\ [\mu(s)^2\lambda(s)^2 + 18\eta(s)\mu(s)\lambda(s)\theta - 4\eta(s)\lambda(s)^3 - 4\mu(s)^3\theta - 27\eta(s)^2\theta^2]$$

Table 5-1

	$\eta(s)>0$	$\eta(s)<0$
$x_1+x_2+x_3$	$+$	$-$
$x_1x_2+x_2x_3+x_3x_1$	$+$	$-$
$x_1x_2x_3$	$-$ when $\theta>0$ $+$ when $\theta<0$	$+$ when $\theta>0$ $-$ when $\theta<0$

In order to find out the value of (31), let us consider the following equation;

(32) $27\eta(s)^2\theta^2+(4\mu(s)^3-18\eta(s)\mu(s)\lambda(s))\theta-\lambda(s)^2(\mu(s)^2-4\eta(s)\lambda(s))=0$

,and let the zeros of this equation, considering θ as a variable, be $\underline{\theta}$ and $\overline{\theta}$ $(\underline{\theta}<\overline{\theta})$, respectively. When $\eta(s)<\frac{\mu(s)^2}{4\lambda(s)}$, the relation of zeros $\underline{\theta}$, $\overline{\theta}$ becomes $\underline{\theta}<0<\overline{\theta}$, and when $\eta(s)\geqq\frac{\mu(s)^2}{4\lambda(s)}$, it is $\underline{\theta}<\overline{\theta}<0$. Therefore, it is necessary for the study of the structure of vector fields to consider the following three cases one by one as shown in Table 5-2 and 5-3, which are obtained from Table 5-1 by a trivial calculation.

Table 5-2

	① $\theta<\underline{\theta}$	② $\theta=\underline{\theta}$	③ $\underline{\theta}<\theta<0$	④ $\theta=0$	⑤ $0<\theta<\overline{\theta}$	⑥ $\theta=\overline{\theta}$	⑦ $\overline{\theta}<\theta$
Case 1 $\eta(s)<0$	$x_1<0$	$x_1<0$ $x_2=x_3>0$	$x_1<0<x_2<x_3$	$x_1<0$ $x_2=0$ $x_3>0$	$x_1<x_2<0<x_3$	$x_1=x_2<0$ $x_3>0$	$x_1>0$
Case 2 $0<\eta(s)<\frac{\mu(s)^2}{4\lambda(s)}$	$x_1>0$	$x_1=x_2>0$ $x_3>0$	$0<x_1<x_2<x_3$	$x_1=0$ $0<x_2<x_3$	$x_1<0<x_2<x_3$	$x_1<0$ $x_2=x_3>0$	$x_1<0$

Table 5-3

	① $\theta<\underline{\theta}$	② $\theta=\underline{\theta}$	③ $\underline{\theta}<\theta<\overline{\theta}$	④ $\theta=\overline{\theta}$	⑤ $\overline{\theta}<\theta<0$	⑥ $\theta=0$	⑦ $0<\theta$
Case 3 $\eta(s)\geqq\frac{\mu(s)^2}{4\lambda(s)}$	$x_1>0$	$x_1=x_2>0$ $x_3>0$	$0<x_1<x_2<x_3$	$x_1>0$ $x_2=x_3>0$	$x_1>0$	$x_1=0$	$x_1<0$

Case 1: $\eta(s)<0$. In this case D_{II} in (31) becomes non-negative when θ takes a value between $\underline{\theta} \leqq \theta \leqq \bar{\theta}$ where $\underline{\theta}<0<\bar{\theta}$, so that we have three real zeros, each value of which depends on the value of θ. When $\theta<\underline{\theta}$ or $\theta>\bar{\theta}$, D_{II} becomes negative, so that we have only one real zero, the others being imaginary. These relations are precisely given in Table 5-2, from which we can easily illustrate vector fields as in Figure 5-1.

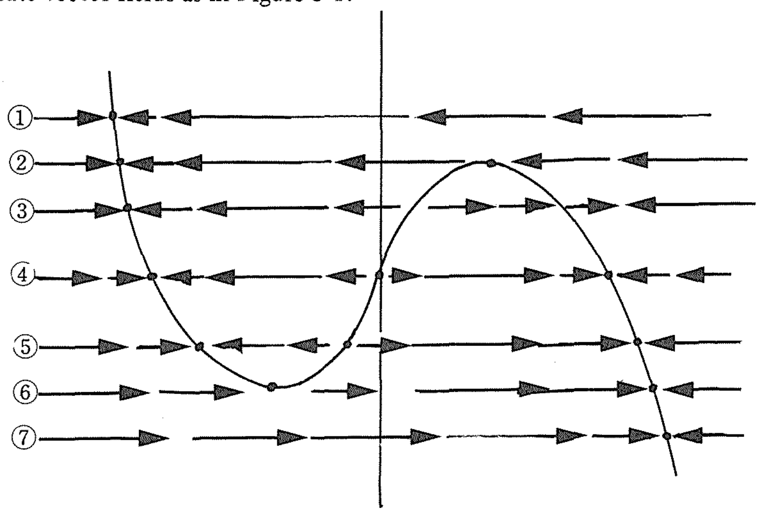

Figure 5-1

These vector fields are further collected into the stracturally different three ones in Figure 5-2.

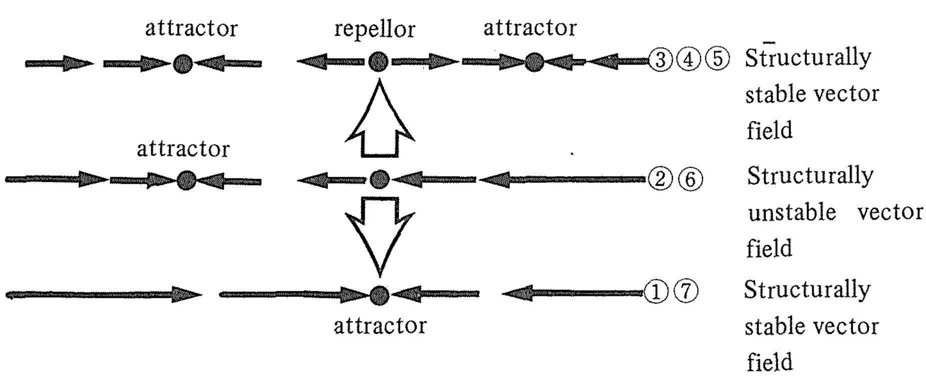

Figure 5-2

We cannot find any vector field near ② or ⑥ in the sense of C^1-norm which is structurally the same, so that ② and ⑥ are structurally unstable and belong to a bifurcation set. $\underline{\theta}$ and $\bar{\theta}$ are clearly included in a catastrophe set.

Case 2: $0 < \eta(s) < \frac{\mu(s)^2}{4\lambda(s)}$. In this case we can illustrate vector fields as in Figure 5-3 by the same reasoning as in case 1.

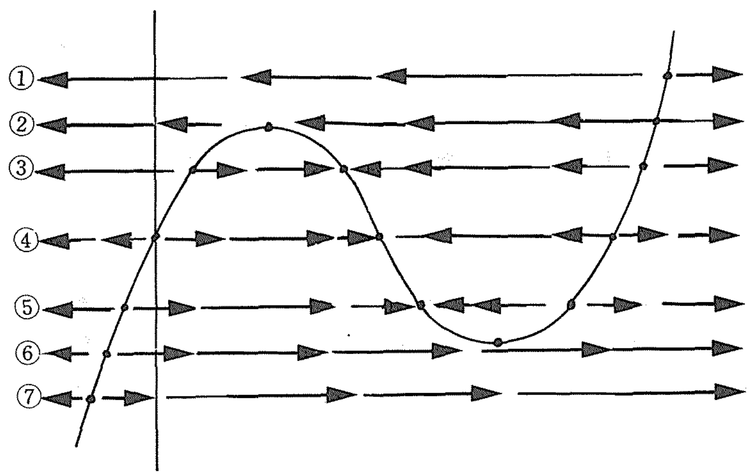

Figure 5-3

We have three structurally different vector fields where ② and ⑥ are structurally unstable vector fields in a bifurcation set.

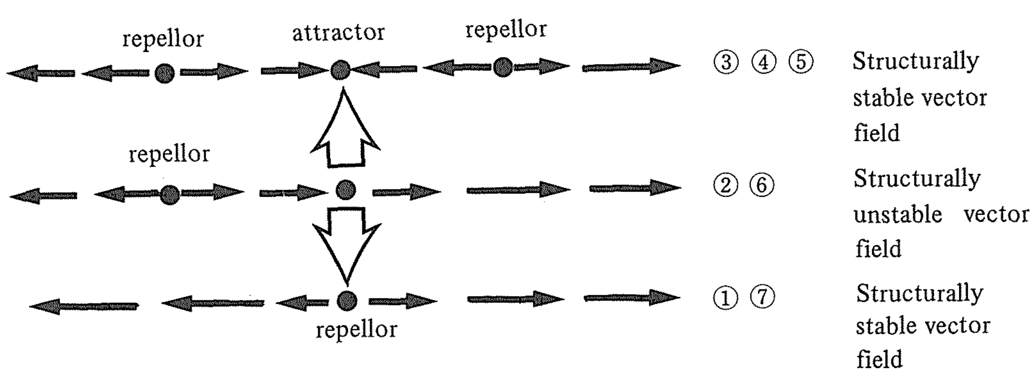

Figure 5-4

Case 3: $\eta(s) \geqq \frac{\mu(s)^2}{4\lambda(s)}$. D_{II} in (31) takes a non-negative value when θ is in the region $\underline{\theta} \leqq \theta \leqq \overline{\theta} < 0$. Hence we have three real zeros depending on the value of θ. When $\theta < \underline{\theta}$ or $\theta > \overline{\theta}$, D_{II} becomes negative and we have only one real zero, the others being imaginary. These relations are precisely given in Table 5-3, from which Figure 5-5 can be illustrated.

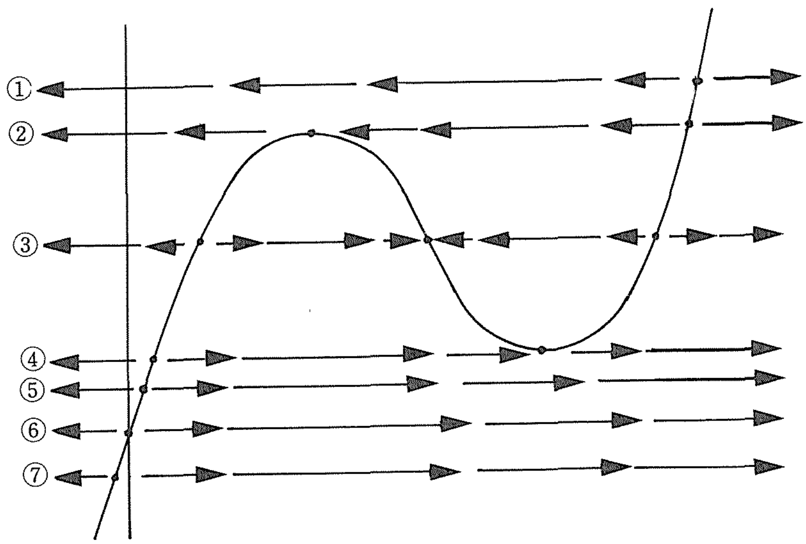

Figure 5-5

The structure of vector fields in this case is the same as in case 2. The only difference is that both of the catastrophe points $\underline{\theta}$ and $\overline{\theta}$ take negative values in this case compared with $\underline{\theta} < 0$ and $\overline{\theta} > 0$ in case 1 and case 2.

We can understand these relations more vividly by looking at the singularities or stationary points in the (x, θ) plane. Figure 5-6 is obtained from the equation;

(33) $\theta = -x(\eta(s)x^2 + \mu(s)x + \lambda(s))$

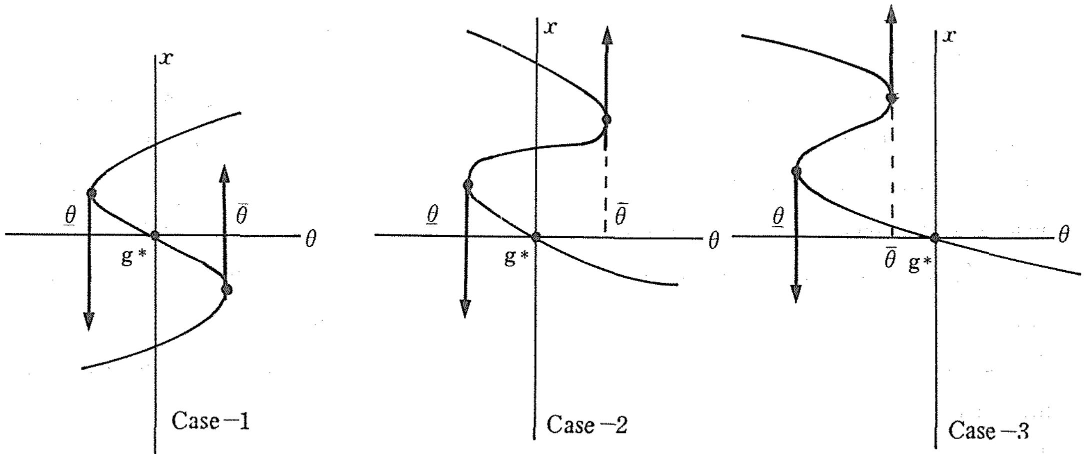

Figure 5-6

Following points are easily derived from the above analysis:

(1) g* always stays in these cases, too, on the unstable surface in the Lyapunov sense.

(2) Case 2 and case 3 can be regarded as structurally the same as the structure in section 3 in the neighborhood of g*; that is, when we start from g*, $\underline{\theta}$ can be the only critical point that brings a catastrophe in the economy.

(3) Case 1 represents a new structure which can be separated out for the first time by expanding the function $\phi(\alpha^{*}-\alpha)$ to the degree three. Both $\underline{\theta}$ and $\bar{\theta}$ bring a catastrophe in the economy. Figure 5-7 suggests the possibility to use this case as a metaphorical model to describe a business cycle. Economic interpretation can be given similarly as in the section 4.

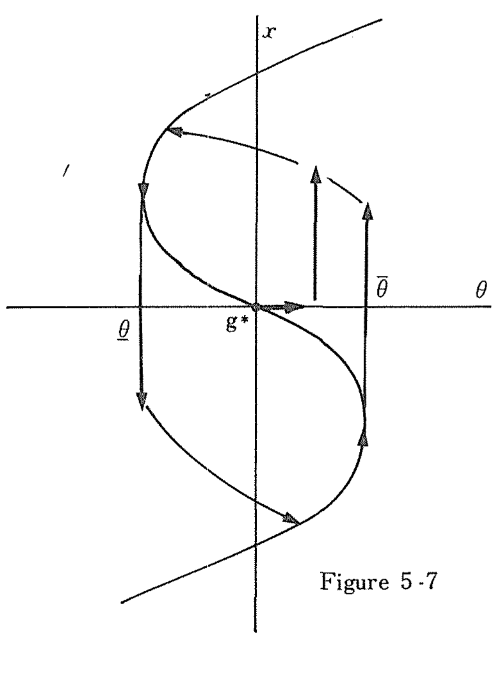

Figure 5-7

6. A metaphor

We can reasonably conclude that the structure in the neighborhood of g* will remain unchanged as in Figure 5-7 even if we continue the expansion of the function $\phi(\alpha^*-\alpha)$ to the infinitely higher degrees and carefully examine the vector fields so long as we confine our analysis to the vector field between the two adjacent attractors to the origin. And it is enough for our purpose to consider the neighborhood of g* only. Samuelson states the similar result: "Likewise, if the system possesses first-order instability, it must be unstable in the small."[8]

Our economy stands on an overall unstable surface. It is easily thrown away from the surface by a small perturbation. The displacement continues to accumulate its unstable degrees. Its one-way movement can only be reversed structurally by the capricious minds of entrepreneurs corresponding to a catastrophe set into the opposite direction all of a sudden. Our economy cannot be free from the violence of business cycle at all even if our government behaves so cleverly. Isn't this a reasonable metaphor to the actual discontinuous phenomenon of business cycle?

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